Full Rank

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In linear algebra we talk about the row and column spaces. Consider an $n \times m$ matrix M. Then its row space is the span of its rows in $\mathbb{R}^m$, and its column space is the span of its columns in $\mathbb{R}^n$. It is clear that the row space of M is the column space of M^T. If we think of M as a linear transformation by multiplication on the left from $\mathbb{R}^m \to \mathbb{R}^n$ then we claim that its column space is isomorphic to the image of this operator.

Proof. Clearly the columns are contined in the image as

$$Me_i = M_i$$

the for $e_i = (0, ..., 0, 1, 0, ..., 0)$ and M_i is the i-th column of M.

Now if $Mu = v \in \text{Im}(M)$ then expressing u in the basis e_i tells us that

$$M(\sum \lambda_i e_i) = \sum \lambda_i M e_i = \sum \lambda_i M_i$$

and so any element is a linear combination of the columns.

The row space is then clearly the image of M^T then. But the rank of a matrix is the same as the rank of its inverse and hence there is an isomorphism between the column space and the row space. Moreover computing the rank is the same as computing the dimension of the image.

Recall that a matrix is called "full rank" if $\text{rank}M = \min(n, m)$, that is the maximum that it could be. Full rank therefore means that the image is as large as it can be, so if $n \leq m$ it would in particular imply that the function is surjective. Note that the rank is connected to the dimension of the kernel, the so called nullity, by the rank nullity theorem

$$\text{rank} + \text{nullity} = m$$

The point is that linear maps can only take spaces to smaller spaces, never bigger.